MATH 323: Algebraic Topology

Exam 2023

Thursday, June 29th

9:15 - 12:15

All your answers need to be justified and explained carefully, unless indicated otherwise. You are allowed to use all the results from the course and the exercises, but you need to indicate clearly, when you do so. It is indicated behind each question how many points it gives. There is a total of 100 points.

As always, S^n denotes the n-sphere

$$S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\},$$

and D^n the closed n-disk

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

Good luck!!!

1. For each of the following statements, state whether it is true (T) or false (F). No justification is needed. You obtain 3 points for each correct answer, 0 points for not answering a question, and -3 points for every incorrect answer. In total, you will not obtain less than 0 points for this question. [**30 points**]

   (a) The n-skeleton of a CW complex X is closed in X.

   True: seen in the course.

   (b) There exists an orientable connected compact manifold of dimension 2023 whose Euler characteristic is 2023.

   False: seen in the exercises with the help of the universal coefficient theorem and Poincaré Duality that an orientable connected compact manifold of odd dimension has Euler characteristic 0.

   (c) Let X be a CW complex and $A \subset X$ a non-empty subcomplex. Then $H_k(X, A) \simeq \tilde{H}_k(X/A)$ for all $k \geq 0$.

   True: CW pairs are good pairs. The statement then follows from a theorem seen in the course.

   (d) Let M be an n-dimensional manifold such that $H_1(M) = 0$. Then M is orientable.

   True: If $H_1(M) = 0$, then the abelianization of $\pi_1(M)$ is trivial. In particular, $\pi_1(M)$ does not contain a subgroup of index 2. We have seen that this implies that M is orientable.

   (e) Let $h\colon S^2 \to S^5$ be an embedding. Then $\tilde{H}_i(S^5 \setminus h(S^2)) = 0$ for all $i \geq 0$ such that $i \neq 2$.

   True: Seen in the course.

   (f) The degree of a surjective map $f\colon S^n \to S^n$ is always non-zero.

   False: We constructed surjective maps of degree 0 in the course.

   (g) Let X be a topological space and $A \subset X$ a retract. Then the homomorphisms $H_k(A) \to H_k(X)$ induced by the inclusion are injective for all $k \geq 0$.

   True: Let $\varphi\colon X \to A$ be a retraction map, i.e., $\varphi(X) = A$ and $\varphi|_A = \mathrm{id}_A$. This implies that φ is a left-inverse of the inclusion $\iota\colon A \to X$. By functoriality we obtain that $\varphi_* \iota_* = \mathrm{id}_{H_k(A)}$ and therefore that ι_* is injective.

(h) Let $n \geq 1$ and let $f\colon S^{2n} \to S^{2n}$ be a continuous map such that $f(x) = f(-x)$ for all $x \in S^{2n}$, then f is of degree 0.

True: seen in the course. This i because $f \circ (-\mathrm{id}) = f$ implies that $-\deg(f) = \deg(f)$.

(i) Let X be a topological space and $A \subset X$, then $H_0(X, A) = 0$ if and only if A intersects each path component of X non-emptily.

True: seen in the course.

(j) Let X be a finite CW complex whose number of n-cells is exactly k. Then $H_k(X) \simeq \mathbb{Z}^k$.

False: Consider for instance S^1 with a CW structure admitting 2 0-cells.

2. Let $U, V \subset \mathbb{R}^n$ be two open subsets such that $U \cup V = \mathbb{R}^n$. Assume that $U \cap V$ has only finitely many path-components

   (a) Show that U and V have only finitely many path-components. [**5 points**]

   Solution: Recall that the Mayer-Vietoris secquence states the following: Let X be a topological space and $A, B \subset X$ two subsets whose interiors cover X, then we have the following exact sequence:

$$\cdots \to H_n(A \cap B) \xrightarrow{\varphi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(A \cup B) \to$$

$$H_{n-1}(A \cap B) \to \cdots \to H_0(A \cup B) \to 0,$$

   where φ is given by $\varphi([\gamma]) = (i_*[\gamma], -j_*[\gamma])$ and ψ is given by $\psi([\alpha], [\beta]) = k_*[\alpha] + l_*[\beta]$. Here, $i \colon A \cap B \to A$, $j \colon A \cap B \to B$, $k \colon A \to X$, $l \colon B \to X$ are the inclusion maps.

   If we assume moreover that $A \cap B \neq \emptyset$ we have the reduced version of Mayer-Vietoris exact sequence:

$$\cdots \to \tilde{H}_n(A \cap B) \xrightarrow{\varphi} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{\psi} \tilde{H}_n(A \cup B) \to$$

$$\tilde{H}_{n-1}(A \cap B) \to \cdots \to \tilde{H}_0(A \cup B) \to 0.$$

   In our case, $U \cap V \neq \emptyset$ since $\mathbb{R}^n$ is connected. So we obtain an exact sequence

$$\tilde{H}_1(\mathbb{R}^n) \to \tilde{H}_0(U \cap V) \xrightarrow{\varphi} \tilde{H}_0(U) \oplus \tilde{H}_0(V) \xrightarrow{\psi} \tilde{H}_0(\mathbb{R}^n).$$

   Since $\tilde{H}_1(\mathbb{R}^n) = \tilde{H}_0(\mathbb{R}^n) = 0$, we obtain that $\tilde{H}_0(U \cap V)$ and $\tilde{H}_0(U) \oplus \tilde{H}_0(V)$ are isomorphic. We have seen in the course that $\tilde{H}_0(X) \simeq \mathbb{Z}^r$ if and only if X has exactly $r+1$ path components. In particular, if $U \cap V$ has only finitely many path-components, then $\tilde{H}_0(U) \oplus \tilde{H}_0(V)$ has finite rank, hence $\tilde{H}_0(U)$ and $\tilde{H}_0(V)$ have both finite rank, whcih implies that U and V have only finitely many path-components.

   (b) Let α be the number of path-components of U, β the number of path-components of V and γ the number of path-components of $U \cap V$. Show that $\gamma = \alpha + \beta - 1$. [**5 points**]

Solution: Using the same argument as in Part (a) of this questions, we find that $\tilde{H}_0(U\cap V)$ and $\tilde{H}_0(U)\oplus\tilde{H}_0(V)$ are isomorphic. Since

$$\operatorname{rank}(\tilde{H}_0(U)\oplus\tilde{H}_0(V)) = \operatorname{rank}(\tilde{H}_0(U)) + \operatorname{rank}(\tilde{H}_0(V)),$$

and moreover

$$\operatorname{rank}(\tilde{H}_0(U)) = \alpha-1, \operatorname{rank}(\tilde{H}_0(V)) = \beta-1, \operatorname{rank}(\tilde{H}_0(U\cap V)) = \gamma-1,$$

the equality

$$\operatorname{rank}(\tilde{H}_0(U\cap V)) = \operatorname{rank}(\tilde{H}_0(U)) + \operatorname{rank}(\tilde{H}_0(V))$$

implies the desired formula.

(c) Assume that $U\cap V$ is simply connected. What can we say about $H_1(U)$ and $H_1(V)$? Justify your answer. **[10 points]**

Solution: We apply the (reduced or non-reduced) Mayer-Vietoris sequence to obtain an exact sequence

$$\tilde{H}_2(\mathbb{R}^n) \to \tilde{H}_1(U\cap V) \xrightarrow{\varphi} \tilde{H}_1(U)\oplus\tilde{H}_1(V) \xrightarrow{\psi} \tilde{H}_1(\mathbb{R}^n).$$

Since $\tilde{H}_2(\mathbb{R}^n) = \tilde{H}_1(\mathbb{R}^n) = 0$, we have $\tilde{H}_1(U\cap V) \simeq \tilde{H}_1(U)\oplus\tilde{H}_1(V)$. If $U\cap V$ is simply connected, then $\tilde{H}_1(U\cap V) = H_1(U\cap V) = 0$, since $H_1(U\cap V)$ is the abelianization of $\pi_1(U\cap V)$. Hence, $H_1(U) = \tilde{H}_1(U) = \tilde{H}_1(V) = H_2(V) = 0$.

(d) Assume that two points $x, y \in U\cap V$ can be connected by a path in U and by a path in V. Can x and y also be connected by a path in $U\cap V$? Justify your answer. **[10 points]**

Solution: Yes, they can also be connected by a path in $U\cap V$. We have $H_1(\mathbb{R}^n) = 0$, hence the Mayer-Vitoris exact sequence implies that the map $\varphi\colon H_0(U\cap V)\to H_0(U)\oplus H_0(V)$ given by $\varphi([\gamma]) = (i_*[\gamma], -j_*[\gamma])$ is injective. Let $[\gamma_x], [\gamma_y] \in H_0(U\cap V)$ be the classes represented by the constant maps at x and y respectively. Since x and y are connected by a path in U and by a path in V, we have that $i_*([\gamma_x]-[\gamma_y]) = j_*([\gamma_x]-[\gamma_y]) = 0$. Injectivity of φ now yields $[\gamma_x]-[\gamma_y] = 0$ and hence that x and y are connected by a path in $U\cap V$.

3. Let $\mathbb{C}P^n$ be the complex projective n-space and let $A \subset \mathbb{C}P^n$ be a subset consisting of 10 points. Compute the relative homology groups $H_k(\mathbb{C}P^n, A)$ for all $k \geq 0$. [**10 points**]

Solution: We have seen that $H_k(\mathbb{C}P^n) \simeq \mathbb{Z}$ if $k \leq 2n$ is even and $H_k(\mathbb{C}P^n) = 0$ if k is odd or $k > 2n$. The long exact sequence for redcued relative homology groups gives us the exact sequence:

$$\cdots \to \tilde{H}_k(A) \to \tilde{H}_k(\mathbb{C}P^n) \to \tilde{H}_k(\mathbb{C}P^n, A) \to \tilde{H}_{k-1}(A) \to$$

$$\cdots \to \tilde{H}_0(\mathbb{C}P^n, A) \to 0.$$

We also know that $\tilde{H}_k(A) = 0$ if $k \geq 1$ and $\tilde{H}_0(A) \simeq \mathbb{Z}^9$ (since A has 10 path-components). This implies that $H_k(\mathbb{C}P^n, A) \simeq \tilde{H}_k(\mathbb{C}P^n, A) \simeq \tilde{H}_k(\mathbb{C}P^n) \simeq \mathbb{Z}$ if $2n \geq k > 1$ and k even, $H_k(\mathbb{C}P^n, A) \simeq \tilde{H}_k(\mathbb{C}P^n, A) \simeq H_k(\mathbb{C}P^n) = 0$ if $k > 2n$ or if $k > 1$ is odd. At the end of the sequence we obtain

$$\tilde{H}_1(\mathbb{C}P^n) = 0 \to \tilde{H}_1(\mathbb{C}P^n, A) \to \tilde{H}_0(A) \to \tilde{H}_0(\mathbb{C}P^n) \to$$

$$\tilde{H}_0(\mathbb{C}P^n, A) \to 0.$$

This implies that $H_1(\mathbb{C}P^n, A) = \tilde{H}_1(\mathbb{C}P^n, A) \simeq \tilde{H}_0(A) \simeq \mathbb{Z}^9$ and $H_0(\mathbb{C}P^n, A) \simeq \tilde{H}_0(\mathbb{C}P^n, A) = 0$.

4. Let A be a finitely generated abelian group. Let A' be the quotient of A by its torsion part (i.e., the quotient of A by its subgroup of elements of finite order). Then $A' \simeq \mathbb{Z}^r$ for some r. A homomorphism $\varphi \colon A \to A$ induces a homomorphism $\varphi' \colon A' \to A'$. If we identify A' with $\mathbb{Z}^r$, then φ' is given by left-multiplication with an integer matrix M. We define the *trace* of φ to be the trace of M. Since the trace is invariant under conjugation, the trace of f does not depend on the choice of isomorphism between A' and $\mathbb{Z}^r$.

Let X be a finite CW complex and $f \colon X \to X$ a continuous self-map. For each $i \geq 0$, the map f induces a homomorphism $f_* \colon H_i(X) \to H_i(X)$. Let t_i be the trace of this homomorphism. We define the *Lefschetz number* $\Delta(f)$ of f as

$$\Delta(f) = \sum_{i \geq 0} (-1)^i t_i.$$

We say that a finite CW complex X has the *Lefschetz property* if the following holds: If a continuous map $f: X \to X$ has no fixed points, then $\Delta(f) = 0$.

The *Lefschetz fixed point theorem* states that every non-empty finite Δ-complex satisfies the Lefschetz property.

Answer the following questions by using these definitions and observations (you do not need to reprove the facts explained in this introductory paragraph).

(a) Show that the Lefschetz number of the identity map is equal to the Euler characteristic of X. [**7 points**]

Solution: The quotient of $H_i(X)$ by its torsion part is isomorphic to $\mathbb{Z}^r$, where $r = \operatorname{rank}(H_i(X))$, by definition of the rank. Let id *be the identity map. Then $\mathrm{id}_*: H_i(X) \to H_i(X)$ is the identity homomorphism, by functoriality. Hence, also the induced map $\mathbb{Z}^r \to \mathbb{Z}^r$ is the identity homomorphism. This implies that the trace of id_* is exactly $r = \operatorname{rank}(H_i(X))$. Therefore, $\Delta(\mathrm{id}) = \sum_i (-1)^i \operatorname{rank}(H_i(X))$, which is exactly the Euler characteristic of X.*

(b) Use the Lefschetz fixed point theorem to show that if X is a contractible finite Δ-complex and $f: X \to X$ a continuous map, then f has a fixed point. [**6 points**]

Solution: If X is contractible, then $H_0(X) \simeq \mathbb{Z}$ and $H_k(X) = 0$ for all $k \geq 1$. Therefore, f_* induces the trivial map on $H_k(X)$ if $k \geq 0$ and f_* induces the identity map on $H_0(X)$, hence the Lefschetz number of f is 1. The Lefschetz fixed point theorem implies that f has a fixed point.

(c) Use the Lefschetz fixed point theorem to show that every continuous map $f: \mathbb{R}P^2 \to \mathbb{R}P^2$ has a fixed point, where $\mathbb{R}P^2$ denotes the real projective plane. [**7 points**]

Solution: We have seen in the course that $\mathbb{R}P^2$ is a finite Δ complex and that $H_0(\mathbb{R}P^2) \simeq \mathbb{Z}$ and $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2\mathbb{Z}$. This implies that the trace of $f_*: H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2)$ is 0, whereas the trace of $f_* H_0(\mathbb{R}P^2) \to H_0(\mathbb{R}P^2)$ is 1. Again, $\Delta(f) \neq 0$ and the Lefschetz fixed point theorem implies that f has a fixed point.

5. (a) Construct a topological space X satisfying the following conditions (justify your answer):

   - $H_0(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$,
   - $H_1(X) = 0$,
   - $H_2(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

   [**5 points**]

   Solution: Consider for example the disjoint union $X = S^2 \coprod S^2$. We have seen that $H_k(A \coprod B) \simeq H_k(A) \oplus H_k(B)$ for all k, if A and B are two non-empty topological spaces. Moreover, we have seen that $H_k(S^n) \simeq \mathbb{Z}$ if $k = 0$ or $k = n$ and $H_k(S^n) = 0$ in all the other cases. Therefore, $H_k(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$ if $k = 0$ or $k = 2$ and $H_k(X) = 0$ else.

   (b) What are the cohomology groups $H_k(X; \mathbb{Z})$? (justify your answer) [**5 points**]

   Solution: By the universal coefficient theorem for cohomology, we have that $H^n(X; \mathbb{Z}) \simeq Free(H_n(X)) \oplus T(H_{n-1}(X))$, where $Free(H_n(X))$ denotes the free abelian part of the finitely generated abelian group $H_n(X)$ and $T(H_{n-1}(x))$ denotes the torsion part of the finitely generated abelian group $H_{n-1}(X)$. Since the torsion part of our homology groups is trivial, we obtain in this case that $H^k(X; \mathbb{Z}) \simeq H_k(X)$ for all k.